

L_2 Residue Separation

A Framework for Spillover Avoidance in PDE Control

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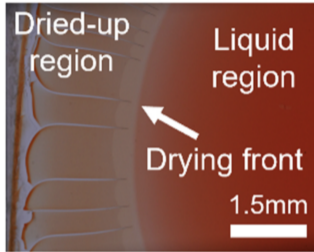
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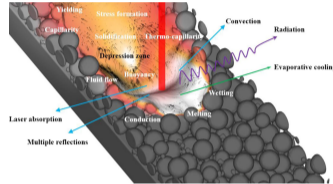
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PDE control in engineering practice

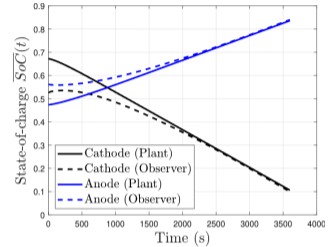
Colloidal thin-film drying¹



Additive manufacturing²



Battery SoC estimation³



Common approach: Replace the PDE by a finite-dimensional approximation and design controllers in that setting.

¹ K. Hatakeyama *et al.*, “Measurement of volume fraction distribution in a drying film by imaging with a digital camera”, *Colloids and Surfaces A: Physicochemical and Engineering Aspects*, 2022.

² M. Bayat *et al.*, “A review of multi-scale and multi-physics simulations of metal additive manufacturing processes with focus on modeling strategies”, *Additive Manufacturing*, 2021.

³ R. Drummond and S. R. Duncan, “Observer design for the Doyle–Fuller–Newman Li-ion battery model without electrolyte dynamics”, *Journal of Energy Storage*, 2019.

Modal Decomposition

Consider the heat equation

$$z_t(x, t) = z_{xx}(x, t) + qz(x, t) + f(\cdot, t, z(\cdot, t))$$

$$z_x(0, t) = 0$$

$$z_x(\pi, t) = u(t)$$

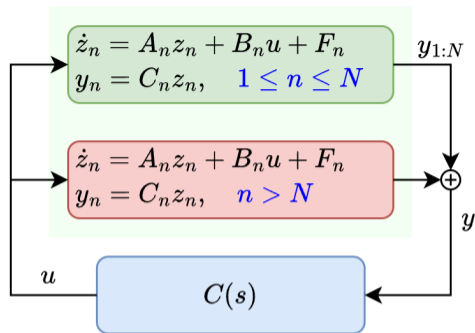
Modal decomposition

$$z(x, t) = \sum_{n=1}^{\infty} z_n(t)\varphi_n(x), \quad \text{where} \quad \begin{cases} \varphi_n''(x) = -\lambda_n\varphi_n(x) \\ \varphi_n'(0) = 0 = \varphi_n'(\pi) \end{cases}$$

This yields infinitely many ODEs

$$\dot{z}_n(t) = -(\lambda_n - q)z_n(t) + f_n(t) + b_n u(t), \quad n \in \mathbb{N},$$

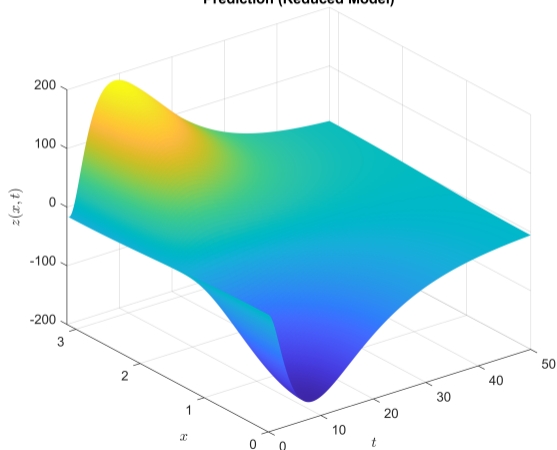
which are then approximated by truncation.



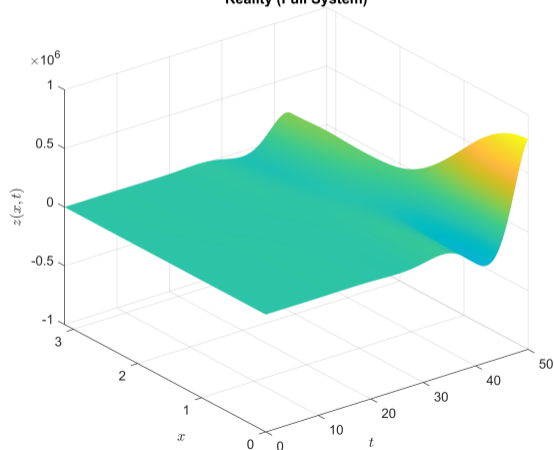
Spillover from neglected modes

$$z_t(x, t) = z_{xx}(x, t) + 1.1z(x, t), \quad z_x(0, t) = 0, \quad z_x(\pi, t) = u(t), \quad y(t) = z(0, t).$$

Prediction (Reduced Model)



Reality (Full System)



Spillover avoidance in modal decomposition

Qualitative results

- ▶ Residual mode filters [Balas'88, Moheimani'98, Harkort & Deutscher'11]
- ▶ Modal decomposition works in practice [Halim & Moheimani'02, Belyaev et al.'18]
- ▶ H_∞ control robust to the residue $\|\Delta\|_\infty$ [Özbay'93, Bontsema & Curtain'88]

Quantitative results

- ▶ LQR to design u benign for the residue (invertible and bounded B) [Hagen & Mezić'03]
- ▶ Lyapunov analysis with Young's inequality for residue separation [Katz & Fridman'20, Karafyllis'21, Lhachemi & Prieur'22]

New concept: L^2 separation of the residue.

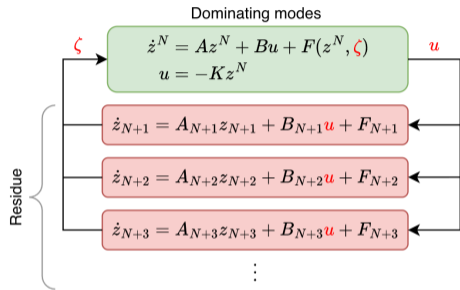
Outline

1. Core idea: The L^2 residue separation
2. Reduced conservatism in guaranteed-cost control
3. Structural insights
 - Improved synthesis of dynamic controllers
 - Gain sparsity in linear systems
 - No loss of admissible nonlinearities for higher-order controllers
 - Sample-and-hold control without lifting
4. Main challenge: computation of L^2 gains
 - Spatially varying coefficients
 - Robin boundary actuation
 - In-domain sensing
 - Hyperbolic and higher-order PDE
5. Beyond Lipschitz nonlinearities: Sum-of-Squares

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The L^2 residue gain(s)



$$\|z_{N+1}(\cdot)\|_{L^2}^2 \leq \gamma_{N+1}^2 \|u(\cdot)\|_{L^2}^2$$

$$\|z_{N+2}(\cdot)\|_{L^2}^2 \leq \gamma_{N+2}^2 \|u(\cdot)\|_{L^2}^2$$

$$\|z_{N+3}(\cdot)\|_{L^2}^2 \leq \gamma_{N+3}^2 \|u(\cdot)\|_{L^2}^2$$

$$\vdots$$

$$\sum_{n=N+1}^{\infty} \|z_n(\cdot)\|_{L^2}^2 \leq \underbrace{\left[\sum_{n=N+1}^{\infty} \gamma_n^2 \right]}_{\gamma^2} \|u(\cdot)\|_{L^2}^2$$

The residue is

$$\zeta(x, t) = \sum_{n=N+1}^{\infty} z_n(t) \varphi_n(x) \quad \implies \quad \|\zeta(\cdot)\|_{L^2(0, \pi) \times L^2(0, \infty)}^2 \leq \gamma^2 \|u(\cdot)\|_{L^2}^2.$$

Main concept: The L^2 separation of the residue

1. Mode-wise control-to-residue gains

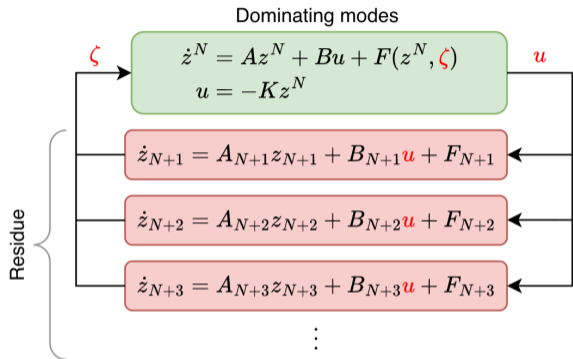
$$V_n = z_n^2, \quad \dot{V}_n - \gamma_n^2 u^2 \leq 0, \quad n > N.$$

2. Aggregated control-to-residue gain

$$\gamma^2 = \sum_{n=N+1}^{\infty} \gamma_n^2.$$

3. Finite-dimensional LQR design

$$V_0 = (z^N)^\top P z^N, \quad \dot{V}_0 + \gamma^2 u^2 \leq 0.$$



Composite Lyapunov functional

$$V = V_0 + \sum_{n=N+1}^{\infty} V_n \quad \implies \quad \dot{V} = \underbrace{[\dot{V}_0 + \gamma^2 u^2]}_{\leq 0} + \sum_{n=N+1}^{\infty} \underbrace{[\dot{V}_n - \gamma_n^2 u^2]}_{\leq 0} \leq 0$$

S-procedure for the nonlinearity

Sector condition on f via its Fourier coefficients F_n :

$$\|f(t, z)\|_{L^2}^2 \leq \sigma^2 \|z\|_{L^2}^2 \quad \implies \quad \sum_{n=1}^{\infty} F_n^2(t) \leq \sigma^2 \sum_{n=1}^{\infty} z_n^2(t)$$

The sector condition introduces the nonlinearity as a quadratic constraint.

S-procedure augmentation

$$\begin{aligned} \dot{V} &\leq \dot{V} + \underbrace{\sigma^2 \sum_{n=1}^{\infty} z_n^2 - \sum_{n=1}^{\infty} F_n^2}_{\geq 0} \\ &= \underbrace{[\dot{V}_0 + \gamma^2 u^2 + \sigma^2 |z^N|^2 - |F|^2]}_{\leq 0} + \sum_{n=N+1}^{\infty} \underbrace{[\dot{V}_n - \gamma_n^2 u^2 + \sigma^2 z_n^2 - F_n^2]}_{\leq 0} \end{aligned}$$

The nonlinearity does not destroy the separation!

Example: Spillover-free control of the heat equation

Semilinear heat PDE with boundary control

$$z_t(x, t) = z_{xx}(x, t) + qz(x, t) + f(x, t, z(\cdot, t)),$$

$$z_x(0, t) = 0, \quad z_x(\pi, t) = u(t).$$

Modal decomposition

$$\dot{z}_n(t) = -(\lambda_n - q)z_n(t) + b_n u(t) + f_n(t)$$

1. Mode-wise control-to-residue gains

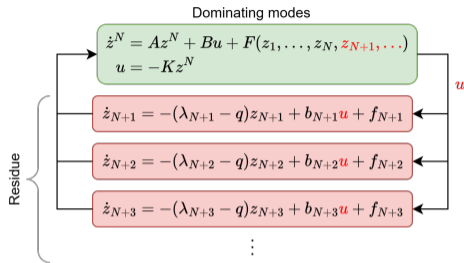
$$\gamma_n^2 = \frac{1}{\pi} \frac{1}{n^2 - q - \sigma}, \quad n^2 > q + \sigma.$$

2. Aggregated control-to-residue gain

$$\gamma^2 = \frac{1}{\pi} \sum_{n=N+1}^{\infty} \frac{1}{n^2 - q - \sigma} \quad (\text{Mittag-Leffler expansion})$$

3. Finite-dimensional LQR design

$$PA + A^\top P + P(\sigma I - \gamma^{-2} BB^\top)P + \sigma I = 0, \quad u = -\gamma^{-2} B^\top P z^N.$$



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Reduced conservatism in guaranteed-cost control

$$\begin{aligned}z_t &= z_{xx} + qz + f(\cdot, t, z(\cdot, t)) \\z_x(0, t) &= 0, \quad z_x(\pi, t) = u(t)\end{aligned}$$

Objective: Find a finite-dimensional state-feedback guaranteeing

$$J = \int_0^\infty [\|z(\cdot, t)\|^2 + ru^2(t)] dt \leq \alpha \|z(\cdot, 0)\|^2.$$

	Minimum guaranteed-cost bound α					
N	1	2	3	4	5	6
L^2 separation	2261.4	521.6	425.9	402.4	393.8	390.1
Young's inequality	25419.2	699.9	480.8	428.2	408.4	399.1

For $N = 1$, the guaranteed cost bound is reduced by more than 90%.

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Output feedback and harmonic inequality

Reaction–diffusion PDE with boundary sensing

$$z_t(x, t) = z_{xx}(x, t) + qz(x, t) + f(x, t, z(\cdot, t)),$$

$$z_x(0, t) = 0, \quad z_x(\pi, t) = u(t), \quad y(t) = z(0, t).$$

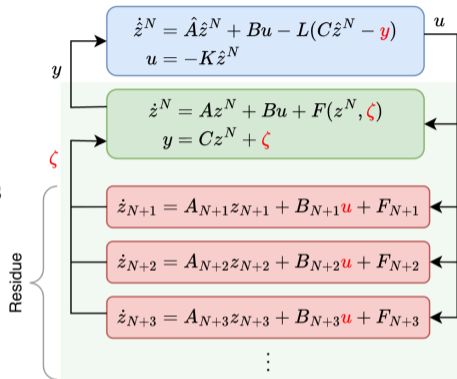
Key issue: Output is contaminated by residual modes

$$y(t) = Cz^N(t) + \zeta(t),$$

$$\zeta(t) = \sqrt{\frac{2}{\pi}} \sum_{n=N+1}^{\infty} z_n(t).$$

Harmonic inequality bounds the output residue:

$$\zeta^2(t) \leq \frac{2}{\pi} \sum_{n>N} \mu_n z_n^2 \quad \text{for } \mu_n > 0 \text{ such that } \sum_{n>N} \mu_n^{-1} \leq 1.$$



Output-feedback design

The H_∞ -inspired controller structure

$$\dot{\hat{z}}^N(t) = (A + \gamma\sigma X)\hat{z}^N(t) + Bu(t) - L(C\hat{z}^N(t) - y(t))$$

$$u(t) = -K\hat{z}^N(t)$$

Note: Instead of $F(\hat{z})$, we incorporate the “worst-case” linear surrogate $\tilde{F} = \gamma\sigma X\hat{z}^N(t)$.

The gains are defined by

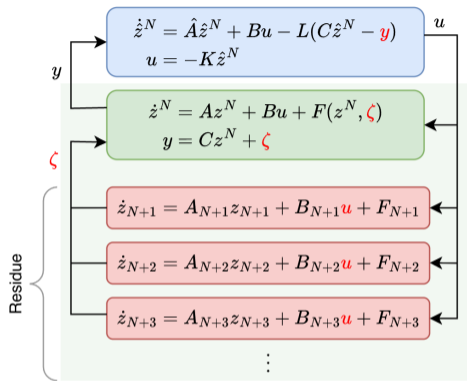
$$K = B^\top X \quad \text{and} \quad L = Z(I_N - \gamma^2 XZ)^{-1}C^\top$$

where $X > 0$ and $Z > 0$ are stabilising solutions of

$$XA + A^\top X - X(BB^\top - \gamma\sigma I_N)X + \frac{\sigma}{\gamma}I_N = 0$$

$$ZA^\top + AZ - Z(C^\top C - \gamma\sigma I_N)Z + \frac{\sigma}{\gamma}I_N = 0$$

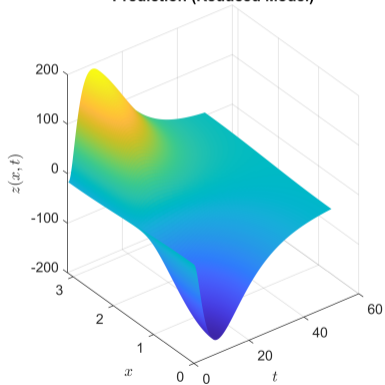
$$\rho(XZ) < \gamma^{-2} \quad (\text{spectral radius condition})$$



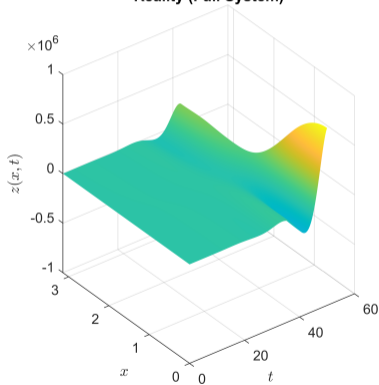
Spillover avoidance via L^2 residue separation

$$z_t(x, t) = z_{xx}(x, t) + 1.1z(x, t), \quad z_x(0, t) = 0, \quad z_x(\pi, t) = u(t), \quad y(t) = z(0, t).$$

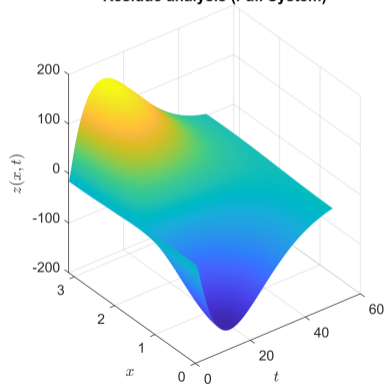
Prediction (Reduced Model)



Reality (Full System)



Residue analysis (Full System)



Structural insights from L^2 residue separation

1. Improved observer design

- ▶ (Unknown) nonlinearity is replaced by a surrogate.
- ▶ Simultaneous synthesis of controller and observer gains.

2. Gain sparsity for linear systems ($f = 0$)

The ARE yields

$$K = [K_0, 0_{1 \times (N-N_0)}], \quad L = \begin{bmatrix} L_0 \\ 0_{(N-N_0) \times 1} \end{bmatrix}, \quad \text{where } \lambda_{N_0} < 0 < \lambda_{N_0+1}.$$

3. Non-decreasing admissibility in N

If the AREs are feasible for some N , they remain feasible for $N + 1$ with the same σ

N	4	5	6	7	8	9
σ	—	0.0019	0.0399	0.0977	0.1611	0.2246
γ	—	0.0456	0.0371	0.0313	0.0271	0.0239

Sample-and-hold control without lifting

$$z_t(x, t) = z_{xx}(x, t) + qz(x, t) + f(t, z(\cdot, t))(x),$$

$$z_x(0, t) = 0, \quad z_x(\pi, t) = u_k, \quad t \in [t_k, t_{k+1}).$$

Standard approach: boundary lifting

$$w(x, t) = z(x, t) + \psi(x)u(t)$$

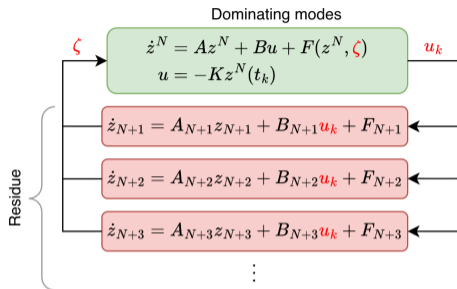
homogenises the boundary conditions but introduces $\dot{u}(t)$, requiring a *generalised hold* implementation.

L^2 residue separation avoids lifting and permits ZOH implementation.

The Lyapunov functional is constructed as

$$V = V_0 + \sum_{n=N+1}^{\infty} V_n + V_h(z_h^N(\cdot)), \quad z_h^N(\theta) = z^N(t + \theta), \quad \theta \in [-h, 0].$$

Key point: the Lyapunov–Krasovskii term V_h depends only on the truncated dynamics and not on the residual modes.



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L^2 residue separation: extensions

1. Spatially-varying coefficients

$$z_t = \frac{1}{w(x)} [p(x)z_x]_x + \frac{q(x)}{w(x)}z + f(t, z), \quad 0 < w, p \in C^2, q \in C.$$

[Petrovsky'54, Orlov'17] prove uniform boundedness of the eigenfunctions:

$$\exists M: \sup_{n \in \mathbb{N}} |\varphi_n(\cdot)|_\infty \leq M.$$

2. Robin boundary conditions

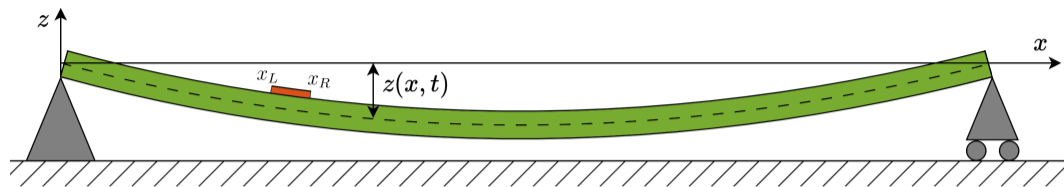
$$\alpha_1 z(0, t) + \alpha_2 z_x(0, t) = 0, \quad \beta_1 z(1, t) + \beta_2 z_x(1, t) = u(t), \quad |\alpha_1| + |\alpha_2| > 0, \quad \beta_2 \neq 0.$$

Dirichlet actuation ($\beta_2 = 0$) may lead to unbounded $b_n = -\frac{p(1)\varphi_n'(1)}{\beta_1}$.

3. In-domain point measurements

$$y(t) = z(x_*, t) = \sum_{n=1}^{\infty} z_n(t) \varphi_n(x_*) \quad \text{with} \quad \sup_{n \in \mathbb{N}} |\varphi_n(x_*)| \leq M.$$

L^2 residue separation for the Euler–Bernoulli beam



The Euler–Bernoulli PDE

$$z_{tt} + z_{xxxx} + c_1 z_t + c_2 z_{xxxxt} = [\delta'(x - x_L) - \delta'(x - x_R)] u + w$$

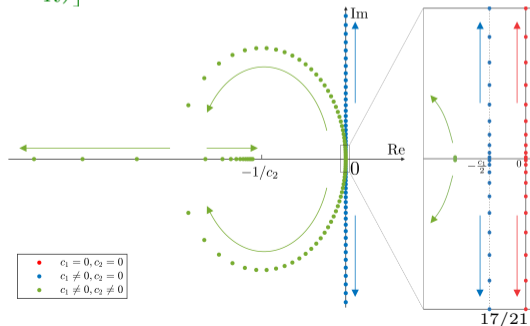
$$z(0, 0) = z_{xx}(0, 0) = 0 = z_{xx}(\pi, 0) = z(\pi, 0)$$

Modal decomposition

$$\ddot{z}_n(t) + 2\zeta_n \omega_n \dot{z}_n(t) + \omega_n^2 z_n(t) = b_n u(t) + w_n(t)$$

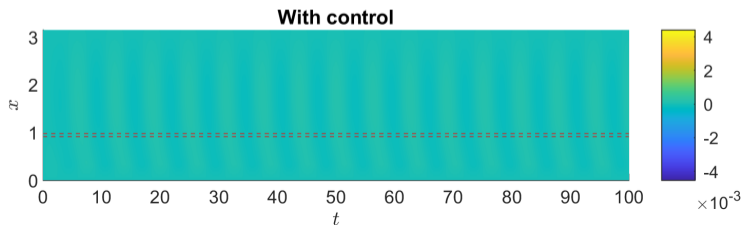
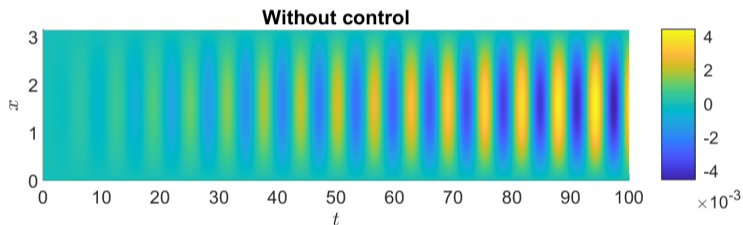
The L^2 gains γ_n computed using BRL with

$$P_n = \frac{2\zeta_n \omega_n}{\alpha_n} \begin{bmatrix} \omega_n^2 & \zeta_n \omega_n \\ \zeta_n \omega_n & 1 \end{bmatrix} > 0.$$



H_∞ control of the Euler–Bernoulli beam

$$J(t) = \int_0^t \left[\|z(\cdot, t)\|^2 + q_x \|z_{xx}(\cdot, t)\|^2 + ru^2(t) - \rho^2 \|w(\cdot, t)\|^2 \right] dt$$



L^2 residue separation for the wave equation

$$\begin{aligned}z_{tt}(x, t) &= z_{xx}(x, t) - 2\mu z_t(x, t) + az(x, t) + f(z(x, t)) \\z(0, t) &= 0, \quad z_x(1, t) = u(t)\end{aligned}$$

	Admissible Lipschitz constant σ_{\max}							
N	1	2	3	4	5	10	15	20
L^2 separation	0.649	1.43	1.92	2.441	2.578	2.734	2.797	2.832
Young's inequality	0.204	0.468	0.676	0.84	0.933	1.066	1.106	1.127

References

- [1] P. Wang, L. Pan, A. Selivanov, and E. Fridman, 'Finite-dimensional boundary control for 1-D semilinear wave equation', IFAC-PapersOnLine, vol. 59, no. 8, pp. 161–166, 2025.
- [2] P. Wang, A. Selivanov, and E. Fridman, 'Sampled-data observer-based boundary control for wave equations with Kelvin-Voigt damping', IFAC World Congress, 2026.

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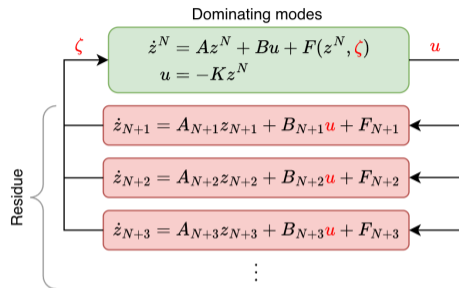
Towards more general nonlinearities

The sector bound

$$\sum_{n=1}^{\infty} F_n^2(t) \leq \sigma^2 \sum_{n=1}^{\infty} z_n^2(t)$$

allows for separation:

$$\begin{aligned} \dot{V} \leq & [\dot{V}_0 + \gamma^2 u^2 + \sigma |z^N|^2 - \sigma^{-1} |F|^2] \\ & + \sum_{n=N+1}^{\infty} [\dot{V}_n - \gamma_n^2 u^2 + \sigma z_n^2 - \sigma^{-1} F_n^2] \end{aligned}$$



Potential extensions

- ▶ Improved DoA for Burgers-type nonlinearity?
- ▶ Improved analysis guaranteeing $\dot{V}_0 + \gamma^2 u^2 < 0$ with a larger $\gamma > 0$?
- ▶ Constraints on f leading to polynomial constraints on $F_n(t) = \langle f(z), \phi_n \rangle_{L^2}$?
- ▶ Improved analysis guaranteeing $\dot{V}_n - \gamma_n^2 u^2 < 0$ with smaller $\gamma_n > 0$?

Summary: L^2 residue separation

Main idea

- ▶ Mode-wise control-to-residue gains: γ_n
- ▶ Aggregated control-to-residue gain: $\gamma^2 = \sum_{n=N+1}^{\infty} \gamma_n^2$
- ▶ Finite-dimensional LQR design minimising $\gamma^2 \|u(\cdot)\|_{L^2}^2$

Key structural insights

- ▶ Improved synthesis of dynamic controllers
- ▶ Gain sparsity in linear systems
- ▶ Non-decreasing admissibility of nonlinearities in N
- ▶ Sample-and-hold control without lifting

Extensions

- ▶ H_∞ control of Euler–Bernoulli beams
- ▶ (Sampled-data) control of the wave equation

